

CONSTRUCTING NON-COMPACT OPERATORS INTO c_0

IRYNA BANAKH AND TARAS BANAKH

ABSTRACT. We prove that for each dense non-compact linear operator $S : X \rightarrow Y$ between Banach spaces there is a linear operator $T : Y \rightarrow c_0$ such that the operator $TS : X \rightarrow c_0$ is not compact. This generalizes the Josefson-Nissenzweig Theorem.

By the Josefson-Nissenzweig Theorem [6], [7] (see also [2], [5, XII], and [3, 3.27]), for each infinite-dimensional Banach space Y the weak* convergence and norm convergence in the dual Banach space Y^* are distinct. This allows us to find a sequence $(y_n^*)_{n \in \omega}$ of norm-one functionals in Y^* that converges to zero in the weak* topology. Such functionals determine a non-compact operator $T : Y \rightarrow c_0$ that assigns to each $y \in Y$ the vanishing sequence $(y_n^*(y))_{n \in \omega} \in c_0$. Thus each infinite-dimensional Banach space Y admits a non-compact operator $T : Y \rightarrow c_0$ into the Banach space c_0 .

The following theorem (which is a crucial ingredient in the topological classification [1] of closed convex sets in Fréchet spaces) says a bit more:

Theorem 1. *For any dense non-compact operator $S : X \rightarrow Y$ between Banach spaces there is an operator $T : Y \rightarrow c_0$ such that the composition $TS : X \rightarrow c_0$ is non-compact.*

By an *operator* we understand a linear continuous operator. An operator $T : X \rightarrow Y$ is *dense* if $T(X)$ is dense in Y .

The proof of Theorem 1 uses the famous Rosenthal ℓ_1 Theorem [8] (see also [5, XI] and [2]) saying that any bounded sequence in a Banach space X contains a subsequence which is either weakly Cauchy or ℓ_1 -basic.

A sequence $(x_n)_{n \in \omega}$ in a Banach space $(X, \|\cdot\|)$ is called ℓ_1 -*basic* if there are constants $0 < c \leq C < \infty$ such that for each real sequence $(\alpha_n)_{n \in \omega} \in \ell_1$ we get

$$c \sum_{n \in \omega} |\alpha_n| \leq \left\| \sum_{n \in \omega} \alpha_n x_n \right\| \leq C \sum_{n \in \omega} |\alpha_n|.$$

Proof of Theorem 1. Assume that $S : X \rightarrow Y$ is a dense non-compact operator. Let $(e_n)_{n \in \omega}$ be the standard Schauder basis of the Banach space c_0 and $(e_n^*)_{n \in \omega}$ is the dual basis in the dual space $c_0^* = \ell_1$. To construct the operator $T : Y \rightarrow c_0$ with non-compact TS , we shall consider three cases.

1. First we assume that the following condition holds:

- (i) there is an ℓ_1 -basic sequence $(y_n^*)_{n \in \omega}$ in Y^* such that the sequence $(S^* y_n^*)_{n \in \omega}$ is ℓ_1 -basic and weak* null in X^* .

In this case we define the operator $T : Y \rightarrow c_0$ by $T : y \mapsto (y_n^*(y))_{n \in \omega}$. Observe that the dual operator $T^* : c_0^* \rightarrow Y^*$ maps the n -th coordinate functional $e_n^* \in c_0^*$ onto y_n^* . Consequently, the sequence

$$(S^* y_n^*)_{n \in \omega} = ((TS)^* e_n^*)_{n \in \omega},$$

being ℓ_1 -basic, is not totally bounded in Y^* , which implies that the dual operator $(TS)^* : c_0^* \rightarrow X^*$ is not compact. By the Schauder Theorem [4, 7.7.], the operator $TS : X \rightarrow c_0$ also is not compact.

2. Assume that the condition (i) does not hold but

- (ii) there is an ℓ_1 -basic sequence $(y_n^*)_{n \in \omega}$ in Y^* whose image $(S^* y_n^*)_{n \in \omega}$ is ℓ_1 -basic in X^* .

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In this case, by [5, Exercise 3(i)] the condition (ii) combined with the negation of (i) imply the existence of an ℓ_1 -basic sequence $(x_n)_{n \in \omega}$ in X whose image $(Sx_n)_{n \in \omega}$ is an ℓ_1 -basic sequence in Y . Arguing as in the proof of Josefson-Nissenzweig Theorem [5, p.223], we can construct a bounded linear operator $T : Y \rightarrow c_0$ such that $TS(x_n) = e_n \in c_0$ for all $n \in \omega$. Since the operator TS is not compact, we are done.

3. Assume that (ii) does not hold. Since the operator S is not compact, its dual $S^* : Y^* \rightarrow X^*$ is not compact too, see [4, 7.7]. This means that the image $S^*(B^*)$ of the closed unit ball $B^* \subset Y^*$ is not totally bounded in X^* . Consequently, the dual ball B^* contains a sequence $(y_n^*)_{n \in \omega}$ whose image $(S^*y_n^*)_{n \in \omega}$ is ε -separated for some $\varepsilon > 0$. The latter means that $\|S^*(y_n^* - y_m^*)\| \geq \varepsilon$ for all $n \neq m$.

By the Rosenthal ℓ_1 Theorem, $(S^*y_n^*)_{n \in \omega}$ contains a subsequence which is either weak Cauchy or ℓ_1 -basic. We lose no generality assuming that the entire sequence $(S^*y_n^*)_{n \in \omega}$ is either weak Cauchy or ℓ_1 -basic.

3a. First we assume that the sequence $(S^*y_n^*)_{n \in \omega}$ is weak Cauchy. Then it is weak* Cauchy and being a subset of the weakly* compact set $S^*(B^*)$ weakly* converges to some point $x_\infty^* \in S^*(B^*)$. Fix any point $y_\infty^* \in B^*$ with $S^*(y_\infty^*) = x_\infty^*$. The density of the operator $S : X \rightarrow Y$ implies the injectivity of the dual operator $S^* : Y^* \rightarrow X^*$. The weak* compactness of the closed unit ball $B^* \subset Y^*$ guarantees that $S^*|_{B^*} : B^* \rightarrow X^*$ is a homeomorphic embedding for the weak* topologies on B^* and X^* . Now we see that the weak* convergence of the sequence $(S^*y_n^*)_{n \in \omega}$ to $S^*y_\infty^*$ implies the weak* convergence of the sequence $(y_n^* - y_\infty^*)_{n \in \omega}$ to zero.

Then the bounded operator $T : Y \rightarrow c_0$, $T : y \mapsto ((y_n^* - y_\infty^*)(y))_{n \in \omega}$, is well-defined. Since the set $\{(TS)^*(e_n^*)\}_{n \in \omega} = \{S^*(y_n^* - y_\infty^*)\}_{n \in \omega}$ is ε -separated, the operator $(TS)^* : c_0^* \rightarrow X^*$ is not compact and hence $TS : X \rightarrow c_0$ is not compact too.

3b. Finally, assume that $(S^*y_n^*)_{n \in \omega}$ is an ℓ_1 -basic sequence in X^* . By Proposition 5.10 [4] (the lifting property of ℓ_1), the sequence $(y_n^*)_{n \in \omega}$ is ℓ_1 -basic in Y^* , which contradicts our assumption that the condition (ii) fails. \square

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DEPARTMENT OF FUNCTIONAL ANALYSIS, YA.PIDSTRYHACH INSTITUTE FOR APPLIED PROBLEMS OF MECHANICS AND MATHEMATICS, NAUKOVA 3B, LVIV, UKRAINE

E-mail address: ibanakh@yahoo.com

INSTYTUT MATEMATYKI, UNIwersYTET HUMANISTYCZNO-PRZYRODNICZY JANA KOCHANOWSKIEGO, KIELCE, POLAND, AND DEPARTMENT OF MATHEMATICS, IVAN FRANKO NATIONAL UNIVERSITY OF LVIV, UNIVERSYTETSKA 1, 79000, LVIV, UKRAINE

E-mail address: t.o.banakh@gmail.com